Closure Properties of Regular Languages

We show how to combine regular languages.
Closure Properties

A set is *closed* under an operation if applying that operation to any members of the set always yields a member of the set.

For example, the positive integers are closed under addition and multiplication, but not division.
Closure under Kleene

**Fact.** The set of regular languages is closed under each Kleene operation.

That is, if $L_1$ and $L_2$ are regular languages, then each of $L_1 \cup L_2$, $L_1 L_2$ and $L_1^*$ is regular.
The easiest approach is to show that the REs for \( L_1 \) and \( L_2 \) can be combined or adjusted to form the RE for the combination language.

Example: The RE for \( L_1L_2 \) is obtained by writing down the RE for \( L_1 \) followed by the RE for \( L_2 \).
Fact. The set of regular languages is closed under complementation.

The complement of language $L$, written $\overline{L}$, is all strings not in $L$ but with the same alphabet.

The statement says that if $L$ is a regular language, then so is $\overline{L}$.

To see this fact, take deterministic FA for $L$ and interchange the accept and reject states.
**Fact.** The set of regular languages is closed under intersection.

One approach: Use de Morgan’s law:

\[ L_1 \cap L_2 = \overline{L_1 \cup L_2} \]

and that regular languages are closed under union and complementation.
Product Construction for Intersection

Each state in the *product* is pair of states from the original machines.

Formally, if $L_1$ is accepted by DFA $M_1$ with 5-tuple $(Q_1, \Sigma, q_1, T_1, \delta_1)$ and $L_2$ is accepted by DFA $M_2$ with 5-tuple $(Q_2, \Sigma, q_2, T_2, \delta_2)$. Then $L_1 \cap L_2$ is accepted by the DFA $(Q_1 \times Q_2, \Sigma, (q_1, q_2), T_1 \times T_2, \delta)$ where $\delta((r, s), x) = (\delta_1(r, x), \delta_2(s, x))$.
Example: Even 0’s and 1’s

Suppose $L_1$ is the binary strings with an even number of 0’s, and $L_2$ the binary strings with an even number of 1’s. Then the FAs for these languages both have two states:

And so the FA for $L_1 \cap L_2$ has four states:
Product Construction for Even 0’s and 1’s
Overview

A regular language is one which has an FA or an RE. Regular languages are closed under union, concatenation, star, and complementation.